Vortex dynamics in a three-state model under cyclic dominance

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The evolution of domain structure is investigated in a two-dimensional voter model with three states under cyclic dominance. The study focus on the dynamics of vortices, defined by the points where the three states (domains) meet. We can distinguish vortices and antivortices which walk randomly and annihilate each other. The domain wall motion can create vortex-antivortex pairs at a rate that is increased by the spiral formation due to cyclic dominance. This mechanism is contrasted with a branching annihilating random walk (BARW) in a particle-antiparticle system with density-dependent pair creation rate. Numerical estimates for the critical indices of the vortex density [$\beta=0.29(4)$] and of its fluctuation [$\gamma=0.34(6)$] improve an earlier Monte Carlo study [K. Tainaka and Y. Itoh, Europhys. Lett. **15**, 399 (1991)] of the three-state cyclic model in two dimensions. [S1063-651X(99)09010-8]

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The self-organizing domain structures in the cyclic variants of the Lotka-Volterra model [1,2] have been extensively investigated because similar spatiotemporal oscillations can appear in chemical reactions as well as in more complex ecological processes. These phenomena can be well studied within the formalism of voter models [3] we will follow henceforth. Different versions of the three-state voter models under cyclic dominance on a square lattice were introduced by Itoh and Tainaka [4–6]. Bramson and Griffeath [7] studied the flux and fixation in cyclic particle systems. The pattern formation in cyclic cellular automata was investigated by Fisch [8]. Using a pair approximation, Frachebourg and Krapivsky [9] have shown that fixation occurs in a cyclic Lotka-Volterra model if the system is started from a random initial state and the number of states exceeds a critical value dependent on the dimension. According to this result, the system tends toward a self-organizing, inhomogeneous state if the number of states is less than 14 on a square lattice. Unfortunately, the theoretical understanding of the mechanism maintaining the inhomogeneous state is still incomplete.

In this work we consider the features of a self-organizing domain structure in a two-dimensional system using the concept of vortices defined for three-state models. Instead of studying the average sizes of the domains, Tainaka and Itoh [5] have determined the average density of vortices ($c=\langle N_v\rangle/L^2$, where N_v denotes the number of vortices in a system with $L\times L$ lattice points). The vortices are points in these domain structures where the three different states (A, B, and C) and the three types of domain walls meet. Evidently, the value of c^{-1} represents roughly the average domain area (size). The investigation of vortex density was strongly motivated by the fact that its determination is much easier than the evaluation of the average domain size.

Figure 1 shows a (three-color) domain structure on the macroscopic scale. It is easy to recognize that two types of vortices may be distinguished as indicated by black and white bullets in the figure. We will call them vortex and antivortex depending on whether we find an ABC or ACB order when going clockwise around the center. A simple rule may be deduced at first glance, namely, the number of vortices is even around a closed domain. More precisely, the vortices and antivortices are alternately located along the closed boundary of a domain. This feature has serious consequences when the motion and collision of vortices is considered.

During the time evolution of a three-color domain structure the vortices move together with the boundaries. In these processes the vortices can collide and annihilate each other. Figure 2 illustrates the typical elementary events (after the transient processes) whose combinations describe all the possible phenomena related to the creation, annihilation, and collisions of the vortices. In the present approach the topological situations where four (or more) domain boundaries meet at a given point are considered as instantaneous events of collisions, pair annihilations, or creations. Evidently, the illustrated elementary events modify the connectivity among the vortices. This connectivity, however, can be modified by either fusion or fission of domains without any change in their constellation.

A numeric analysis of the vortex density was performed

FIG. 1. A typical domain structure for a three-state (color) model. The black bullets with white borders and the white bullets with black borders represent the vortices and antivortices.

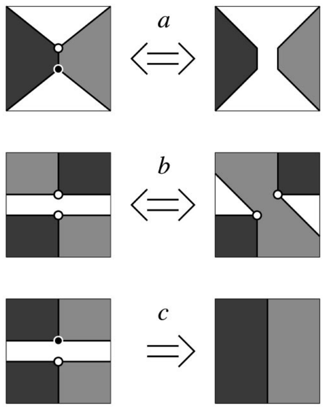

FIG. 2. Schematic plots for characterizing the elements of vortex dynamics on a three-state map. The upper process (a) represents the annihilation of a vortex-antivortex pair walking along their common boundary line as well as the reverse phenomenon corresponding to a spontaneous pair creation. (b) The ''collision'' of two vortices or antivortices can modify the type of domain separating them. (c) A vortex-antivortex pair can annihilate each other in a different way when crossing through the separating domain.

by Tainaka and Itoh [5] in a two-dimensional voter model where the voters, located on a square lattice, should choose among three states (opinions): A, B, and C. The system evolution is governed by a simple algorithm: a randomly chosen voter or one of its nearest neighbors can modify their opinion if those are different. If the chosen voter and its neighbor are in the states A and B, then the first voter adopts opinion B with probability P, otherwise its neighbor changes its state to A. This adoption rule is repeated cyclically for the B-C and C-A cases. Similar cyclic dominance characterizes the ''paper, scissors, stone'' games [10].

The direction of dominance can be reversed by replacing 1-P for P, therefore the analyses are restricted to $P>1/2$. For $P=1/2$ the present model is equivalent to a traditional voter model [3], which exhibits a (three-color) domain coarsening phenomenon if initially the voter states are random. In other words, the finite system evolves into one of the three homogeneous states while the vortex density goes to zero.

For $P>1/2$, however, a self-organizing domain structure is maintained, and the vortex density tends to a stationary value, N_v dependent on P. More precisely, Tainaka and Itoh have found a power law behavior, namely,

$$c \sim (P-P_c)^{\beta}, \tag{1}$$

where $P_c=1/2$ and $\beta \sim 0.40$ [5].

We have repeated these simulations using larger system size (400×400) and longer sampling times. In the vicinity of P_c ($P-P_c<0.01$) the thermalization is chosen to be longer than 250 000 MCS (Monte Carlo steps per particle). In the stationary state we have determined all the four-point configuration probabilities $Q(n_1,n_2,n_3,n_4)[n_i=A,B,$ or C; $(i=1,2,3,4)]$ of a 2×2 cluster. In our notation the (A,A,B,C) configuration refers to a vortex, the (A,A,C,B) to an antivortex, and the (A,B,C,A) configuration can be interpreted as a vortex-antivortex pair (before their annihilation or after their birth). Monitoring the vortex density we were able to determine its average value and its fluctuation $[\chi=L^2\langle(N_v/L^2-c)^2\rangle]$ in the stationary states as a function of $p\equiv P-P_c$.

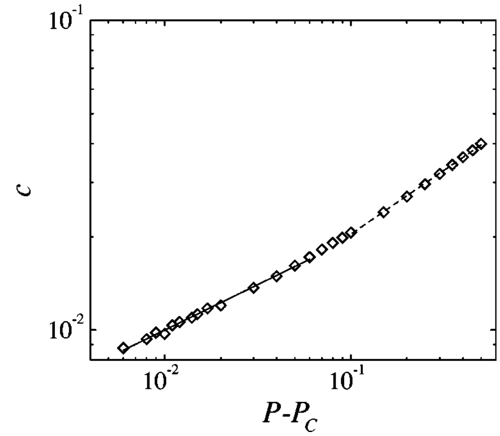

FIG. 3. Log-log plot of vortex density vs P in the three-candidate voter model under cyclic dominance. The open diamonds represent MC data, the solid and dashed lines (resp. slopes 0.29 and 0.41) indicate the fitted power laws.

The results of our simulations are summarized in two log-log plots (see Figs. 3 and 4). Figure 3 demonstrates that we have reproduced Tainaka and Itoh's data [5] for $p>0.1$. In this region the fitted power law (dashed line) is characterized by an exponent $\beta=0.41(3)$ in agreement with Tainaka and Itoh. For smaller p values, however, we have found a different exponent $\beta=0.29(4)$ (solid line). Similar crossover behavior can be observed for the fluctuation as shown in Fig. 4. The fluctuation remains approximately constant for $p>0.1$, whereas it follows a power law $[\chi\sim(P-P_c)^{-\gamma}$ with $\gamma=0.34(6)]$ for smaller p values.

The above-mentioned configuration probabilities satisfy the reflection, rotation of $90°$ and cyclic symmetries. Consequently, all these quantities are describable, within a cluster approximation, by introducing six independent parameters that can be evaluated by solving a set of equations of motion for the four-point configuration probabilities. Previously, this method has been proved to be a very efficient tool

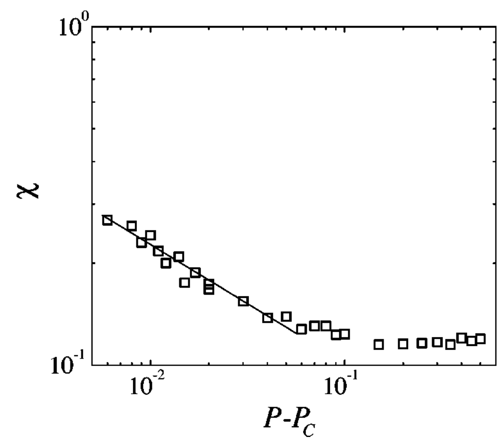

FIG. 4. Log-log plot of density fluctuation vs $P-P_c$ in the three-candidate voter model under cyclic dominance. MC results are represented by open squares, the fitted power law function is represented by a solid line for small $P-P_c$ values.

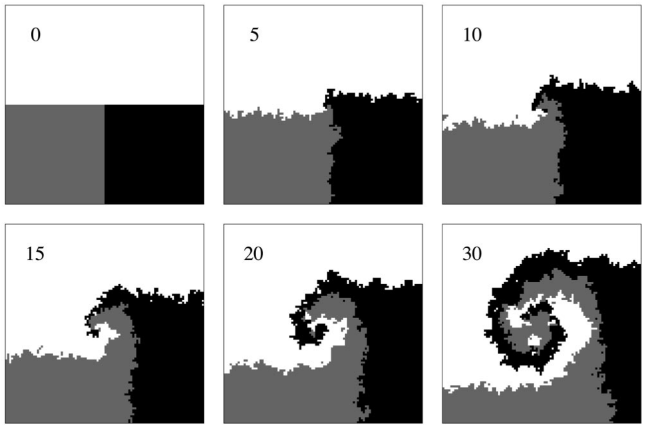

FIG. 5. Time evolution of a vortex initially having straight border lines for $P=1$. The figures at the upper-left corners indicate the time measured in MCS units.

for the investigations of stochastic cellular automata [11], evolutionary games [12], different lattice versions of the Lotka-Volterra models (at the level of pair approximation) [6,9], and for a two-dimensional driven lattice gas maintaining a self-organizing domain structure, too [13]. In the present case, the calculations show a very weak P dependence of the configuration probabilities, including the vortex constellations mentioned above, in contrast with the MC simulations. At the level of the pair approximation the configuration probabilities are independent of P [6], while the mean-field (one-point) approximation predicts homogeneous oscillatory behavior. At the five-point level, on the other hand, the preliminary calculations (integrating numerically the equations of motion) also indicate a very weak P dependence. This puzzling failure of the dynamical cluster technique inspired us to search for a mechanism observable at macroscopic (or mesoscopic) scale.

For $P=1/2$, the domain coarsening is accompanied by the annihilation of pairs, and this process is not prevented by the weak spontaneous pair creation. Under cyclic dominance ($P>1/2$), however, we have detected the appearence of a pair creation mechanism which is able to compensate for the previous annihilation process, yielding a finite vortex density. We have displayed in Fig. 5 the evolution of a single vortex whose geometry allows us to recognize the essential processes. Here, instead of the usual periodic boundary conditions, we have assumed that a voter residing on the perifery and its ''hypothetic'' outer neighbor have always the same opinion. Along the boundaries one can observe a cyclic invasion whose average velocity component perpendicular to the border line is proportional to $P-P_c$. This motion of the boundaries yields a spiral formation around the center. The average time evolution is decorated by noise as shown in a series of snapshots in Fig. 5. Due to the randomness the neighboring boundaries can contact and create vortex-antivortex pairs inside the spiral because it consists of narrow ''arms,'' which is a favored situation for the pair creation. The created pairs can be considered as the offspings of the original vortex. Most of the pairs are annihilated within a short time but, sooner or later, a vortex-antivortex pair will eventually drift apart from each other. The corresponding vortices will then expand and become capable to create further offsprings via the same spiral formation process. The self-organizing domain structure can thus be maintained by this mechanism.

Within the framework of the vortex language, the evolution of domain structure can be described by the two-dimensional motion of vortices, allowing the annihilation and creation of pairs. Assuming that the motion of vortices is dominantly controlled by noise, the present problem can be considered as a so-called branching annihilating random walk (BARW) with two types of particles created and annihilated in pairs. The branching process is evidently controlled by the value of $P-P_c$, though the mathematical relation between the branching rate and $P-P_c$ is not yet clarified. Furthermore, the branching rate is affected by the nearest neighbor distances because it limits the spiral formation.

The traditional BARWs have been intensively studied in the last years (for recent reviews, see the work by Cardy and Täuber [14] and Marro and Dickman [15]) because they undergo a critical transition when varying the rate of branching. The corresponding critical behavior belongs to the directed percolation (DP) universality class [16] involving the Reggeon field theory [17], the surface reaction [18] and Schlögl models [19], and the extinction phenomena observed in spatial evolutionary games [12]. According to the ''DP conjecture'' [20] most of the one-component model with a single absorbing state belong to the DP universality class. Exceptions can appear when additional symmetries [21] or conservation laws are introduced. Well known examples are those models in which the parity of particles is conserved during the elementary processes [22,23]. The introduction of two (or more) types of particles has enlarged the number of possible universality classes [14,24].

The field theoretical results [14,24,25] indicate that mean-field approaches can give satisfactory description of the two-dimensional BARW models with two types of particles. This observation motivated us to try to describe the branching annihilating random walks of vortices and antivortices through a mean-field equation with a density-dependent branching rate, namely,

$$\frac{\partial}{\partial t}c=-c^2+\lambda(c)c, \tag{2}$$

where c denotes the concentration of vortices and antivortices. The first term describes the annihilation process whose prefactor is eliminated by choosing a suitable time scale. For simplicity we suppose that the density dependence of the branching rate follows a power law with exponent ν,

$$\lambda(c)=pc^{-\nu}, \tag{3}$$

where $p=A(P-P_c)$. Notice that this branching rate diverges in the limit $c\to 0$ if $\nu>0$. In the stationary state one can easily determine the concentration as a function of p:

$$c\sim p^{1/(1+\nu)}. \tag{4}$$

Comparing this formula with the above MC results one can conclude that the present mean-field description predicts $\nu \simeq 1.5$ if $P-P_c>0.1$ and $\nu\simeq 2.5$ for the smaller value of $P-P_c$.

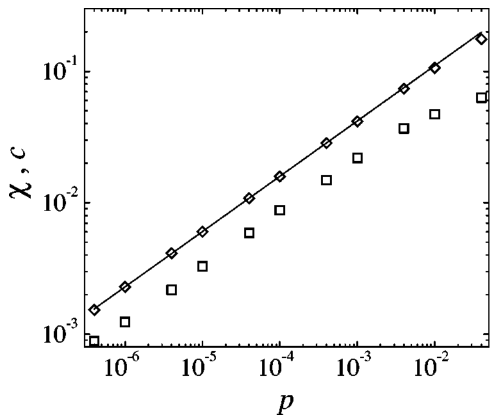

FIG. 6. MC results for the particle-antiparticle concentration (open diamonds) and its fluctuation (open squares) as a function of p in the BARW model described in the text. The solid line (slope 0.42) indicates a fitted power law.

In order to check the role of fluctuations we have performed MC simulations on a particle-antiparticle BARW model. The system evolution is governed by nearest neighbor jumps, particle-antiparticle pair annihilations, and creations as follows. A randomly chosen particle (or antiparticle) can create a particle-antiparticle pair located on two randomly chosen nearest neighbor sites with a probability P_{br}, otherwise this particle jumps to one of the nearest neighbor sites. The processes which would result in two particles (or antiparticles) residing on the same point are blocked. A particle-antiparticle pair is annihilated if they would stay on the same point as a result of the mentioned elementary processes. The branching rate P_{br} is determined for a given particle as a product $P_{br}=pR_1R_2R_3$, where R_1 (R_2, R_3) denotes the distance between the chosen particles and its first (second, third) neighbor antiparticles. The choice of this branching rate is motivated by the topological fact that, in the original voter model, a vortex is connected directly to three antivortices by boundary lines (see Fig. 1). During the simulations the value of P_{br} can become larger than 1 very rarely, thus, we did not need to reduce the time unit in which each particle has a chance to create offsprings or to jump. Initially particles and antiparticles with equal numbers are distributed randomly on a square lattice. The system size is varied from $L=100$ to 400 when decreasing p. During the simulations we have recorded the number of particles and determined the average value of the concentration c and its fluctuation χ in the stationary state as defined above. From these MC results (see Fig. 6) we could confirm that the concentration follows a power law (solid line) with an exponent $\beta=0.42(3)$. It is emphasized that this value of β agrees very well with the prediction of the above mean-field formalism ($\beta_{MF}=0.4$ for $\nu=1.5$). This result implies that any value of the exponent β is reproducible with the parameter adjustment of a more sophisticated BARW model.

Considering the fluctuations obtained by simulations, a striking difference is found between the present BARW model and the three-candidate voter model. The BARW simulations (squares in Fig. 6) indicate that $\chi(p)\propto c(p)$, which seems to be a typical behavior for the BARW models [25,26]. For the voter model, on the contrary, χ diverges (see Fig. 4) for small values of the control paramater. We can identify two possible sources for this discrepancy. First, during the diffusive motion the particle-antiparticle annihilation results in an aggregation of the two species [27], whereas in the vortex-antivortex system such a process is strongly limited by the mentioned topological features. Second, the threefold degeneracy of the absorbing states of the three-candidate voter model has no counterpart in the traditional BARW models. Notice that, in the suggested BARW models, the evolution into an absorbing state is prevented by the divergency of the branching rate in the limit $c\to 0$ for sufficiently large system size. Further systematic research is required to clarify the effect of these phenomena.

To summarize, in the present paper we have improved the accuracy of the numerical analysis of the critical transition appearing in the vortex density for the three-candidate voter model when varying the magnitude of cyclic dominance. Recognizing that the dynamics of the vortices and antivortices is similar to a BARW model with a density-dependent particle-antiparticle pair creation (branching) rate, we have contrasted these two systems. According to our comparison, we can state that the power law behavior of the vortex density is reproducible with a suitable choice of the pair creation mechanism. The same is not true for the behavior of fluctuations, which seems to be quite different in the two models. This discrepancy is a motivation to seek further extensions of BARW models, since this approach seems to be very useful in the investigation of the self-organizing, three-color domain structures.

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[1] A. J. Lotka, Proc. Natl. Acad. Sci. USA **6**, 410 (1920).
[2] V. Volterra, *Lecon sur la Theorie Mathematique de la Lutte pour la Vie* (Gauthier-Villars, Paris, 1931).
[3] T. M. Ligget, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[4] Y. Itoh, Prog. Theor. Phys. **78**, 507 (1987).
[5] K. Tainaka and Y. Itoh, Europhys. Lett. **15**, 399 (1991).
[6] K. Tainaka, Phys. Lett. A **176**, 303 (1993); **207**, 53 (1995).
[7] M. Bramson and D. Griffeath, Ann. Prob. **17**, 26 (1989).
[8] R. Fisch, Physica D **45**, 19 (1990).
[9] L. Frachebourg and P. L. Krapivsky, J. Phys. A **31**, L287 (1998).
[10] For a recent textbook, see P. Morris, *Introduction to Game Theory* (Springer-Verlag, New York, 1994).
[11] H. A. Gutowitz, J. D. Victor, and B. W. Knight, Physica D **28**, 18 (1987); G. Szabó and G. Ódor, Phys. Rev. E **49**, 2764

(1994); H. Rieger, A. Schadschneider, and M. Schreckenberger, J. Phys. A **27**, L423 (1994).

[12] G. Szabó and C. Tőke, Phys. Rev. E **58**, 69 (1998).

[13] G. Szabó and A. Szolnoki, Phys. Rev. E **53**, 2196 (1996); G. Szabó, A. Szolnoki, T. Antal, and I. Borsos, *ibid.* **55**, 5275 (1997).

[14] J. L. Cardy and U. C. Täuber, Phys. Rev. Lett. **77**, 4780 (1996); J. Stat. Phys. **90**, 1 (1998).

[15] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England, 1999).

[16] S. R. Broadbent and J. M. Hammerlslay, Proc. Cambridge Philos. Soc. **53**, 629 (1957); J. Blease, J. Phys. C **10**, 917 (1977); **10**, 923 (1977); **10**, 3461 (1977).

[17] V. N. Gribov, Zh. Éksp. Teor. Fiz. **53**, 654 (1967) [JETP Lett. **26**, 414 (1968)]; H. D. Aberbanel, J. B. Bronzan, R. L. Sugar, and A. R. White, Phys. Rep. **21**, 119 (1975); P. Grassberger and K. Sundermeyer, Phys. Lett. B **77B**, 220 (1978); M. Moshe, Phys. Rep. **37**, 255 (1978); J. L. Cardy and R. L. Sugar, J. Phys. A **13**, L423 (1980).

[18] R. M. Ziff, E. Gulari, and Y. Barshad, Phys. Rev. Lett. **56**, 2553 (1986); G. Grinstein, Z.-W. Lai, and D. A. Browne, Phys. Rev. A **40**, 4820 (1989).

[19] F. Schlögl, Z. Phys. **253**, 147 (1972).

[20] H. K. Janssen, Z. Phys. B **42**, 151 (1981); P. Grassberger, *ibid.* **47**, 365 (1982).

[21] H. Hinrichsen, Phys. Rev. E **55**, 219 (1997).

[22] H. Takayasu and A. Yu. Tretyakov, Phys. Rev. Lett. **68**, 3060 (1992).

[23] N. Menyhárd and G. Ódor, J. Phys. A **28**, 4505 (1995); **29**, 7739 (1996).

[24] M. J. Howard and U. C. Täuber, J. Phys. A **30**, 7721 (1997).

[25] B. P. Lee and J. Cardy, J. Stat. Phys. **80**, 971 (1995).

[26] G. Szabó and M. A. Santos, Phys. Rev. E **59**, R2509 (1999).

[27] D. Toussaint and F. Wilczek, J. Chem. Phys. **78**, 2642 (1983).